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First Name: _____
ID Number:_____

CharisMATHic Analysis

March 30, 2019

Problems and Solutions

1. What is the probability of throwing one dice three times in a row and get the even number after the first fall, the number greater than four after the second fall and the odd number after the last fall?

A) 3/8 **B)** 3/12 **C)** 1/12 **D)** 1/8 **E)** 1/18

Answer C. ½ * ⅓ *½ = 1/12

2. A certain price was increased by 40% and then the new price was decreased by 25%. The same final price would be attained by a single increase of the initial price by:

A) 5% **B)** 10% **C)** 15% **D)** 20% **E)** 25%

Answer A. $1.4 \times 0.75 = 1.05$.

3. If $2^8+2^8+2^8+2^8=2^n$, then $n=$ **A)** 9 **B)** 10 **C)** 16 **D)** 24 **E)** 32

Answer B. We have $4.2^8=2^{10}$.

4. If $x+y+z=2019$, $\frac{x}{y}=\frac{1}{2}$ and $\frac{y}{z}=\frac{111}{170}$, find the value of $z-x$.

A) 678 **B)** 687 **C)** 768 **D)** 786 **E)** 867

Answer B. If $y=222k$, then $x=111k$, $z=340k$, $2019=x+y+z=671k$, hence $k=3$, $x=333$, $z=1020$, $z-x=687$.

5. The parallelogram $ABCD$ has perimeter 144. The angle bisectors of the angles at A and B meet at a point L on the side CD. Find AB.

A)24 B)36 C)45 D)54 E)Another answer

Answer E. The angles DAL, BAL and ALD are equal, so $AD=DL$. Similarly $BC=CL$. If $AD=x$, the perimeter is $2x+x+2x+x=144$, hence $x=24$, $2x=48$.

6. What is the median of the following values: 3, -10, 0, 3, 3, 4, -10, 1, 2

A) -4/9 **B)** 3 **C)** 4/9 **D)** 2 **E)** -10

Answer D. The sorted sequence is -10, -10, 0, 1, 2, 3, 3, 3, 4. This means the median (middle value) is 2.

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M1. (**Do this AFTER completing the exam**) Explain the reference in this meme

This is obviously a reference to the "She believed/sbeve" (https://knowyourmeme.com/memes/she-believed-sbeve) meme but the capitalized letters spell out "SEND MEME". Writing down the "hidden text" or showing understanding of the meme can get you 1 meme point.

7. Four of the angles in a pentagon are equal, while the fifth one is twice as small. Find the measure of the largest angle in this pentagon.

A) 60º **B)** 80º **C)** 100º **D)** 120º **E)** 140º

Answer D. If the angles are $2x$, $2x$, $2x$, $2x$, x, their sum is $9x=540º$, hence $x=60º$, $2x=120º$.

8. How many integers n satisfy the condition $2019\leq|n|\leq9102$?

A) 7083 **B)** 7084 **C)** 14166 **D)** 14168 **E)** Another answer

Answer D. There are $9102–2019+1=7084$ positive numbers and that many negative ones.

9. Based on the 3Blue1Brow Essence of Calculus video series, which of the following definitions best describes the **intuition (**i.e. conceptual understanding) behind a derivative? Try to motivate your choice.

A) The area under a curve **B)** The difference between two values
C) The "acceleration" of a function **D)** The best constant approximation for rate of change
E) A tangent line to some curve

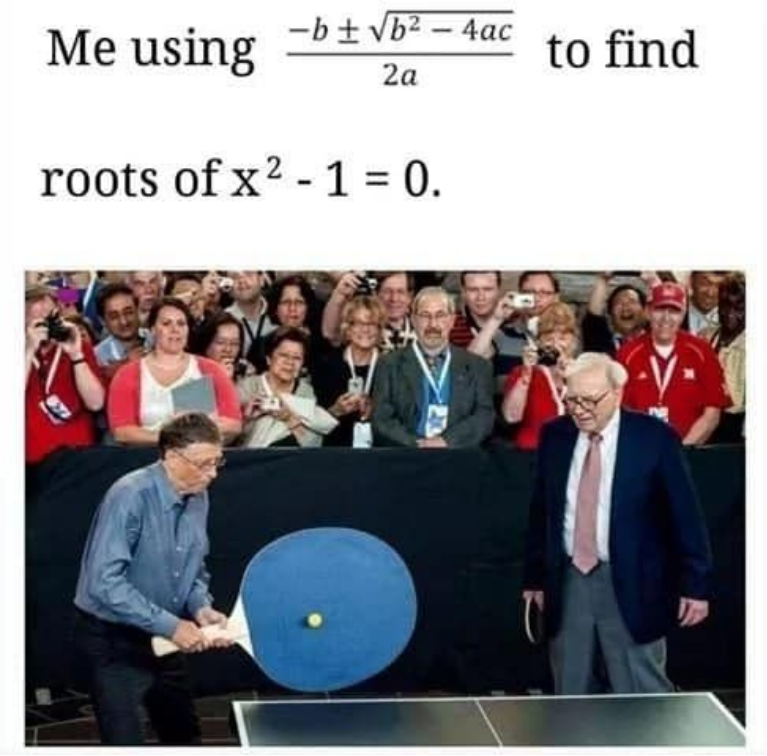
Answer D. Both A and C are not correct (A is the integral and C is the second derivative). B only speaks about any values and not if they are close of getting closer. E is not correct as it is not a tangent like but can be seen as the slope of a tangent line.

10. A teacher has to choose three students from the class 3A and two students from the class 3B to the recitation competition. There are 22 students in the 3A and 17 students in the 3B class. How many possible choices does she have? [C(X, Y) means combinations of X for Y0]

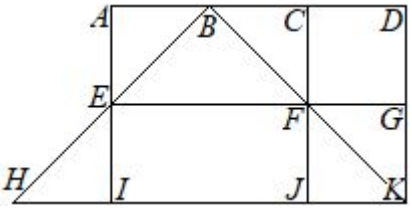
A) C(22,3)*C(17,2) **B)** C(22,3)+C(17,2) **C)** C(39,5) **D)** C(23,3)+C(18,2) **E)** C(23,3)*C(18,2)

Answer A. We first have to choose 3 from 22 (i.e. this is what combinations mean) AND (multiplication of probabilities) then choose 2 from 17. The two choices combinations are obviously independent of each other - every students is in either one of the classes but not in both.

M2. (**Do this AFTER completing the exam**) Do you feel like this now?



This is a reference to the "Bill Gates with the giant ping pong paddle" meme (https://knowyourmeme.com/memes/bill-gates-giant-ping-pong-paddle). It is a joke about students using unnecessary complex methods to solve trivial problems. Example, a lot of participants in previous years wanted to use calculators even for problems where using a calculator would probably slow them down. You can earn a meme point here if you are creative or relate somehow to the meme.



11. The picture shows a map of a village (the lines are the streets). You have to put policemen at three of the intersections (denoted with letters) so that each street is under police surveillance (i.e. has a policeman at some of its intersections). There must be a policeman at which of the following intersections:

A) *A* **B)** *B* **C)** *C* **D)** *D* **E)** It cannot be uniquely determined

Answer C. There must be a policeman on each of the three streets going north to south and a policeman on each of the three streets going east to west. So there is no policeman at *B* nor at *H*, hence the diagonal *BEH* has to be monitored from *E*. So there is no policeman at *F* nor at *G*, hence the diagonal *BFK* has to be monitored from *K*. So there is no policeman at *I* nor at *J*, hence the street *CFJ* has to be monitored from *C*. Now clearly *C*, *E*, *K* form an appropriate choice.

12. Maria is papering a 5×5 wall using seven square wallpapers of identical size, marked by different symbols. No wallpaper goes beyond the borders of the wall. Some of the wallpapers overlap, so six of them are only partially visible. In what order were they placed?

A)  **B)** 

C)  **D)** 

E) None of the above

Answer A. The blue-flowered wallpaper covers more area than any other, so it has been placed last. Therefore all other wallpapers, before being covered, have had its dimensions. The suitcase wallpaper is not covered from the right, so it has to be covered from its left, hence the hearts wallpaper is placed later. The hearts wallpaper is not covered from top, so it has to be covered from bottom, hence the mirrors wallpaper is placed later. The mirrors wallpaper is not covered from top, so it has to be covered from bottom, hence the stars wallpaper is placed later. The stars wallpaper is not covered from left, so it has to be covered from right, hence the leaf wallpaper is placed later. The leaf wallpaper is not covered from left, so it has to be covered from right, hence the basketball wallpaper is placed later.

13. The average weight of five boys is 59kg. The average weight of two of these boys is 56kg. What is the average weight (in kg) of the other three boys?

Answer 61. We have (59.5–56.2)/3=61.

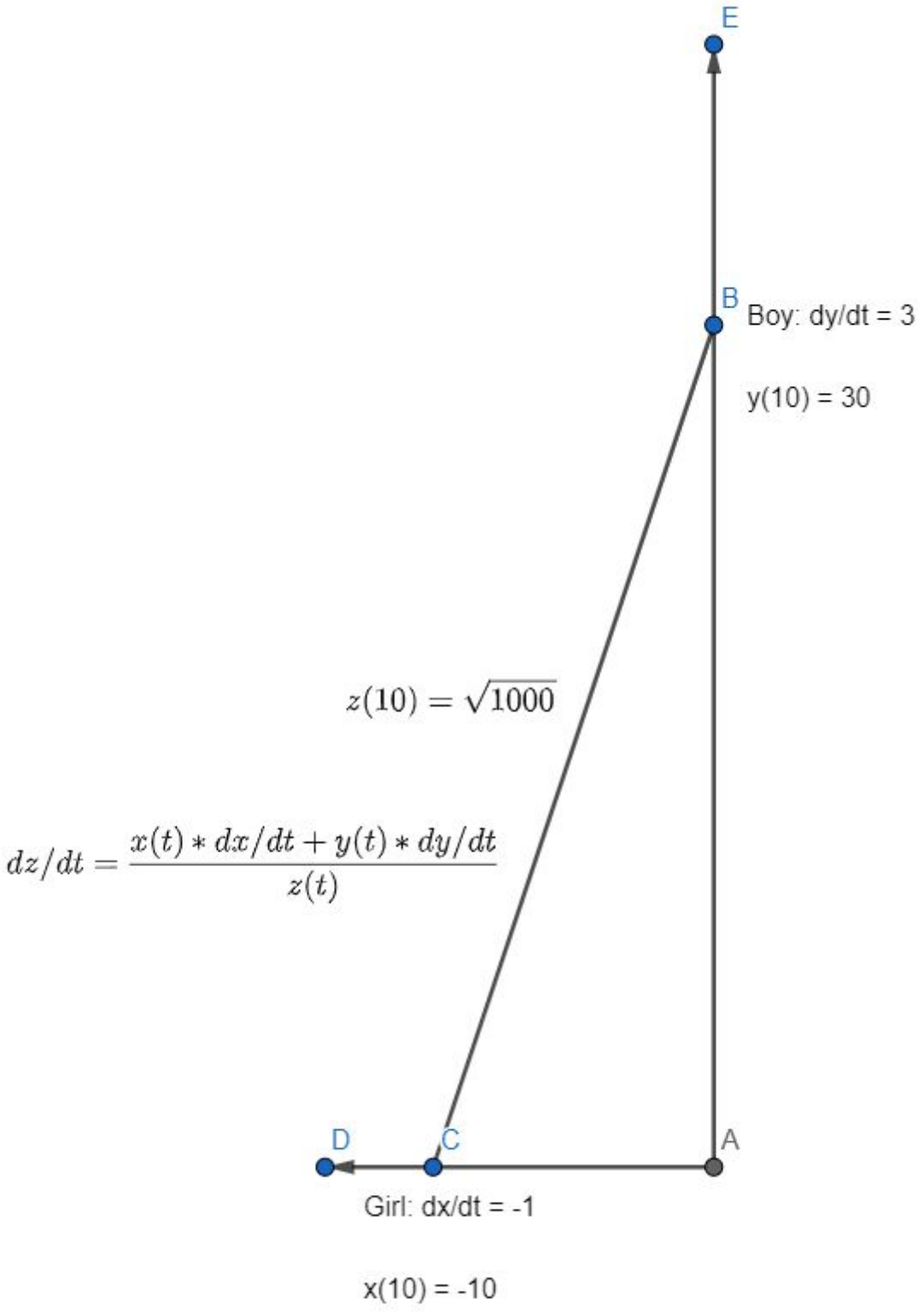
14. One rainy day, a girl broke up with her boyfriend after being together for 5 long years. They decided to part ways where everything about them began at the same time. The boy is due north shaking and crying, running at 3 m/s and the girl is walking due west at a rate of 1 m/s thinking she made the right decision. How fast are they separating from each other after 10 seconds in their new lives without each other?

Hint: This is a related rates problem (i.e. the 6th 3Blue1Brown video).

Answer $\sqrt{10}$**.** As stated this is a related rates problem based on the calculus preparatory videos (specifically, the last one: https://youtu.be/qb40J4N1fa4). Since this problem is a bit more involved, in order to get full credit, all you have to do is (1) show that the difference between the two people is described by the hypotenuse of a right angle triangle which can be found using Pythagoras' theorem and (2) you have to implicitly differentiate the equation with respect to time since the two sides of the triangle are increasing at a given rate, the hypotenuse increases at a rate that can be discovered using implicit differentiation and the chain rule.

Specifically, you have the picture bellow (the way the two are walking away from each other). The relation of the distance between the two of them is $z^2 = x^2 + y^2$ and since we are talking about distances in time, we can think of each of x, y, z as functions of time, i.e. x(t), y(t), z(t). We are given also how x and y change (their derivatives): dx/dt = -1 (girl walking west) (if we take west to mean go left on the graph but we can also take it to mean right, without any loss of meaning) and dy/dt = 3 (boy running north). Then, using what we learn from the video on implicit differentiation we have: $2z(t) * dz/dt = 2x(t) * dx/dt + 2y(t) * dy/dt$. After simplification, we can find the what the derivative of z is: $dz/dt = \frac{x(t)*dx/dt+y(t)*dy/dt}{z(t)}$. All we

have to do now is plug in the proper numbers for when t=10 (10 seconds after they seperate). So we have x(10) = -10, y(10) = 30, dx/dt = -1, dy/dt = 3, $z(10) = \sqrt{x(10)^2 + y(10)^2} = 10\sqrt{10}$. Which will give us: $dz/dt = \frac{10+90}{10\sqrt{10}} = \frac{10}{\sqrt{10}} = \sqrt{10}$ m/s.



15. You have to create numbered cards for a sports competition for participants of your school. You are almost done with that but they **change the format** the very last minute! Each numbered card is **3** digits and people from your school can only use the following numbers: *3, 5, 7, 8, 9, 0* (only once, without repetitions). The resulting number has to be divisible by **5** and **cannot** start with **0**. How many such cards can you create?

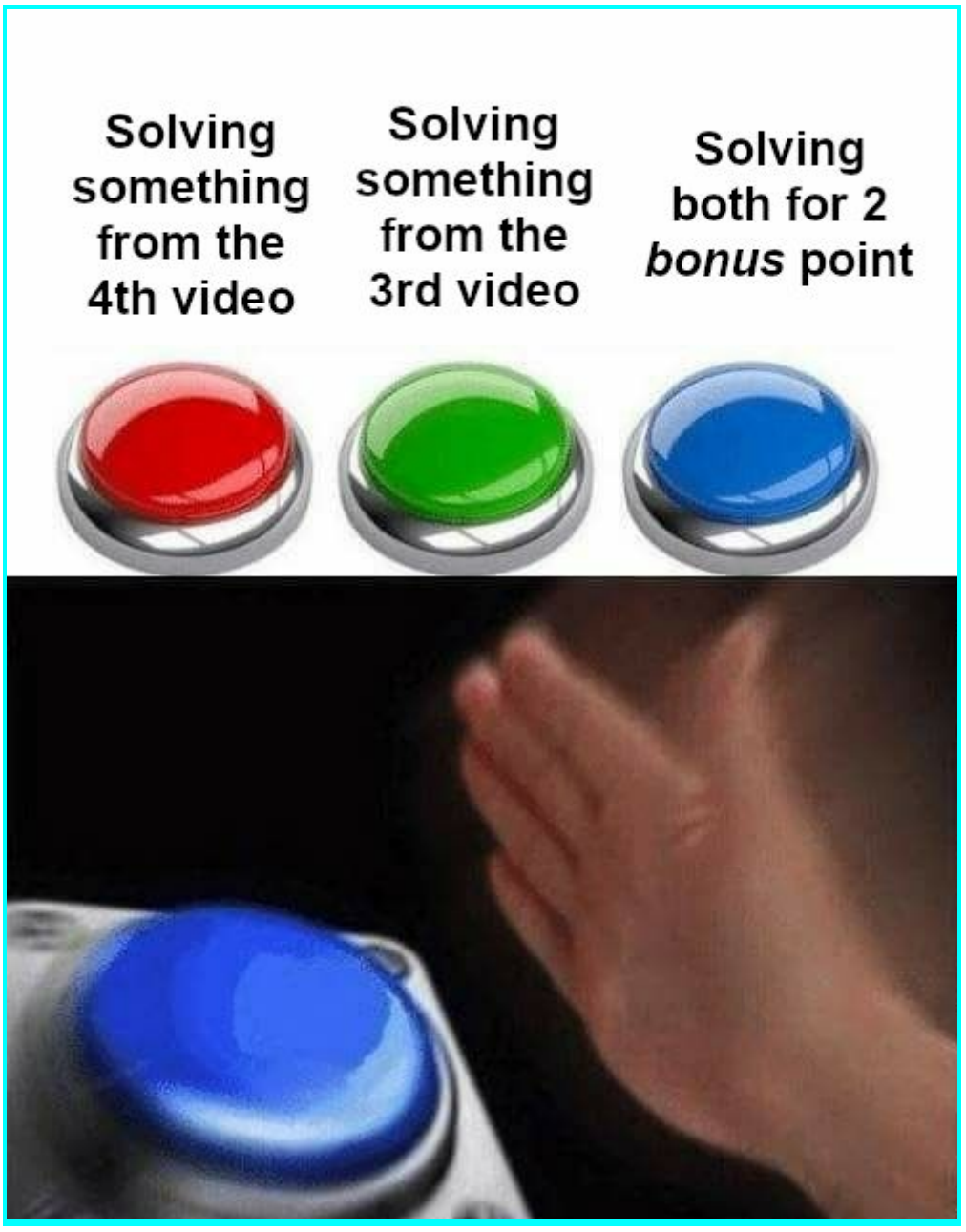
Answer 36. There are two ways to create the given number of cards. The number has to end in 0 or in 5. If it ends in 0 we can choose 5 of the numbers for the first place and 4 for the second. Thus, we have 5*4=20 for the first option. If it ends in 5, we have 4 numbers for the first place (we cannot use 0) and also 4 places in the second place (we can no use zero). Thus, we get 4*4=16 for the second option. Since this is an either or choice, we get 20+16=36 as the final answer.

16. Providing a specific example, that it is possible to raise an irrational number to an irrational power and get a rational solution. Provide the necessary algebraic manipulation to show that this is correct.

Answer (example) $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$ **.** Showing that this is rational is simply a matter of algebraic operations on power functions. When you have a number to some power to some other power a^{b^c}, you can simplify it to a^{b*c}. So this will give you $\sqrt{2}^{\sqrt{2}*\sqrt{2}} = \sqrt{2}^2 = 2^{\frac{1}{2}*2} = 2$. We also have to note that $\sqrt{2}$ is irrational (no need to show that it is irrational) and that for $\sqrt{2}^{\sqrt{2}}$ we have 2 cases:

1) $\sqrt{2}^{\sqrt{2}}$ is rational, which actually proves our problem by default (irrational to an irrational power is rational)
2) $\sqrt{2}^{\sqrt{2}}$ is irrational which means we can use the above algebraic manipulation to show that we can get to a rational number if we use it as a power.

M3. Solver **either** problem *17.a* **or** *17.b*. You will be given **full** credit for **only one**, but you will get 2 extra points if you **solve both.**



This a reference to the "Multiple Choice" button version of the 'Blue Button" meme (https://knowyourmeme.com/memes/nut-button). The description says it all, but the meme implies that you should attempt to some both problems (provided you have watched the 3Blue1Brown videos) to get extra points. Also, this gives you the opportunity to earn an extra meme point if you are creative.

17.a. What is so special about Euler's number - e? Write down what the main algebraic property of that number is with regard to derivatives $d(e^x)/dx$ **and** write down the formula for the derivative of any power function $d(p^x)/dx$ (using the natural logarithm $ln(a)$).

Answer d(e^x)/dx = e^x and d(p^x)/dx = ln(p)*p^x. Euler's number is special is that its exponential function is its own derivative. For the general case, we can reduce any exponential to that of the Euler number.

You can try to also explain what you learned from the videos if you are not sure about the exact answer.

17.b. Write down the general form of the chain rule of derivatives (i.e. for functions of this kind $f(x) = g(h(x))$) **and** find the exact derivative of this function $f(x) = (sin(x))^2$. For the general form, use $d(f(x))/dx$ to mean the derivative of $f(x)$ with respect to x where x can also be a function like h (the derivative with respect to some function.

Answer d(g(h(x)))/dx = d(g(h))/dh(h(x))*dh/dx(x) and 2*sin(x)*cos(x). Simply write down the chain rule - the outer function depends on the inner value and the inner depends only on x.

You can try to also explain what you learned from the videos if you are not sure about the exact answer.

18. Find the largest three-digit number that has the same remainder when divided by 5, 6 and 7.

Answer 844. If x is the number in question and r is the remainder, then $x–r$ is divisible by 5, 6 and 7, hence by 210. The largest three-digit multiple of 210 is 210.4=840 and the largest possible remainder is r=4, hence the answer is x=844.

19. Tom is three years younger than Mary and his age is 80% of hers. How many years from now will the age of Tom be equal to 95% of the age of Mary?

Answer 45. If Mary now is $5x$, Tom is $4x$=$5x$–3, hence x=3, $5x$=15, $4x$=12. Their difference is 3 years and it will be equal to 5% of her age when she is 60. This will happen 60–15=45 years from now.

20. In the right $\triangle ABC$ the altitude CH to the hypotenuse intersects the angle bisector AL at its midpoint O. If OH=2, find BC.

Answer 12. Since CO is a median in a right triangle, we have $CO=AO=LO$. Hence the angles HAO, OAC and ACH are all equal and hence equal to 30º. Therefore angle B is also equal to 30º. Using the property of the 30º-60º-90º triangles, from OH=2 we get consecutively OA=4, OC=4, CH=6, CB=12.

21. Someone comes up and asks you to prove that any square number (of the form a^2) when multiplied by a number that is 17 more than it ($a^2 + 17$), is always a multiple of 6, else she will tell everyone about that one accident during the winter break. You do not want to risk it.

Possible solutions.

Method 1 (Direct)
If a number, x, is divisible by 6, then x-18 will be divisible by 6 too (as 18 is a multiple of 6).
Thus, if x2 (x2-1) divides 6, then x2 (x2+17) will divide it too.
x(x2-1) can be factorized into x(x-1)(x+1), which is the product of three consecutive integers.
That guarantees that among them we have an even number and a number, divisible by 3.
Hence, x(x2-1) is even and divisible by 3, thus divisible by 6.
As x(x2-1) is a factor of x2 (x2-1), then x2 (x2-1) is divisible by 6 and thus x2 (x2+17) is divisible by 6. Q.E.D.

Method 2 (Hipster)
For this proof, nCr will show a combination of r out of n elements. nCr.
("n choose r" or C(n, r)).

Let us consider in how many ways can we choose 4 out of a elements, a>4, positive integer.
Thus we construct the combination aC4.
Combination always gives a positive integer as an output (or positive integers are closed under combination). Therefore, aC4 is an integer.
Applying the formula for combinations, we can represent aC4 as a!/(4!(a-4)!)
This simplifies to aC4=a(a-1)(a-2)(a-3)/24.
Since aC4 is an integer, a(a-1)(a-2)(a-3) is divisible by 24 (and thus 6).
By expanding brackets, we get a4-6a3+11a2-6a is divisible by 6.
Since -6a3-6a clearly divide 6, that means that a4+11a2 also divides 6.
Thus, if we add 6a2 to the expression, it will still divide 6.
We then get that a4+17a2 is divisible by 6, which can be factorized to the required form.
Note: this is a proof for any a>4 (and, since a is squared in the expression, any a<-4). For a=0,1,2,3 we get 0, 18, 84 and 234, all multiple of 6, thus completing the proof.

M4. (**Do this AFTER completing the exam**) Caption this.

(Note, the low resolution is intentional)

This is a reference to the "And that's a fact" meme (https://knowyourmeme.com/memes/and-thats-a-fact**).** Here, you have another opportunity to earn a meme point. You can directly follow the meme and write down something on the blank screen, make some MTQ reference, a Pewdiepie reference or be creative.

22. You risked and failed your English literature exam (Who would have guessed all the answers were false!?!) but you have a makeup tomorrow. It has 8 questions, each of them being true-false. You still feel lazy. You need 40 percent correct answers or better. Calculate the probability of getting a passing score by guessing, again. Are you going to risk it this time?

Solution:

We can divide the problem into several discrete probabilities for getting exactly n problems correctly. In order to pass the test, you should answer at least half of the problems because ⅜=0.375 and you need at least 0.4 to pass. Thus, the probability of us passing is if we get exactly 4, 5, 6, 7 or 8 answers correctly. The "or" means we have to sum the probabilities. Let us look at one specific case:

For getting 4 correctly we need 1⁄2 * 1⁄2 * 1⁄2 * 1⁄2 = 1/16 for guessing true and 1/16 for guessing false, analogously. In order to get the probability of the event, we multiply the number of ways to pick certain number of questions from the test with the probability of guessing respective number of questions correct and false. We can focus on the correct problems only, since the incorrect will be fixed once the correct ones are fixed. We have C(8, 4) ways to choose the correct answers. Thus, we have C(8, 4)*1/16*1/16.

Since we have to sum all the different events (getting 4,5,6,7,8 correct) we simply repeat the above logic and we have:
P("Passing") =
C(8, 4)*1/16*1/16 + C(8, 5)*1/32*1/8 +
C(8, 6)*1/64*1/4 +C(8, 7)*1/128*1/2+C(8, 8)*1/256*1/1

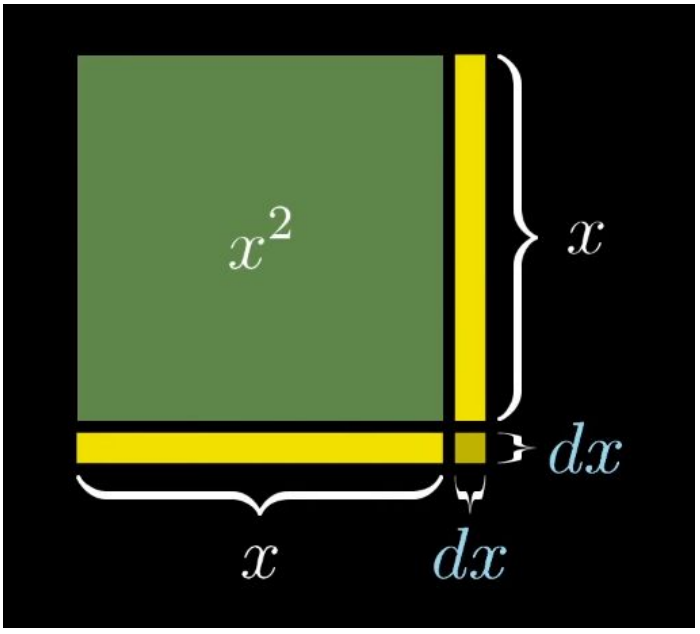
You do not need to calculate this but the end result is **163/256**. Which is 0.63671875, so that is not too bad. You might as well risk it (provided there is another makeup for which you can actually study if you get the unlucky 0.36328125 probability this time).

23. Your friend is about to have a calculus quiz on derivatives and he is completely confused at the concept of derivatives. You decide to channel your inner 3Blue1Brown and help your friend. Give a visual proof, like those found in the videos, of what the derivative of $\sqrt{x}$ is.

Proof.
This is based on the material in the preparatory homework problems. The 3Blue1Brown video where the general idea of this can be found is this: https://youtu.be/S0_qX4VJhMQ.

One way to prove this is to consider a square with an area of x and sides $\sqrt{x}$. This way, you can apply the geometric reasoning found the video of thinking about the derivative as some small change in some geometric figure. In this case, we want to increase the area of the square (while still preserving its property of equal sides - so that it stays a square). Using the intuition in the videos, we increase each side by a tiny amount $d\sqrt{x}$. The area of the parts that contribute to the new area (considering the tiny amount is very small) is just $d\sqrt{x}$ (we neglect term that contain $d\sqrt{x}^2$).

Thus, we get $\frac{dx}{d\sqrt{x}} = 2\sqrt{x}$. Recall the area is just x. However, we are interested in the inverse relation, how the change in area (x) affects the size of the square ($\sqrt{x}$). So, we can invert the relation and get what we are looking for $\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}$, which is the derivative of the square root function.



[Image taken from the 3Blue1Brown video]

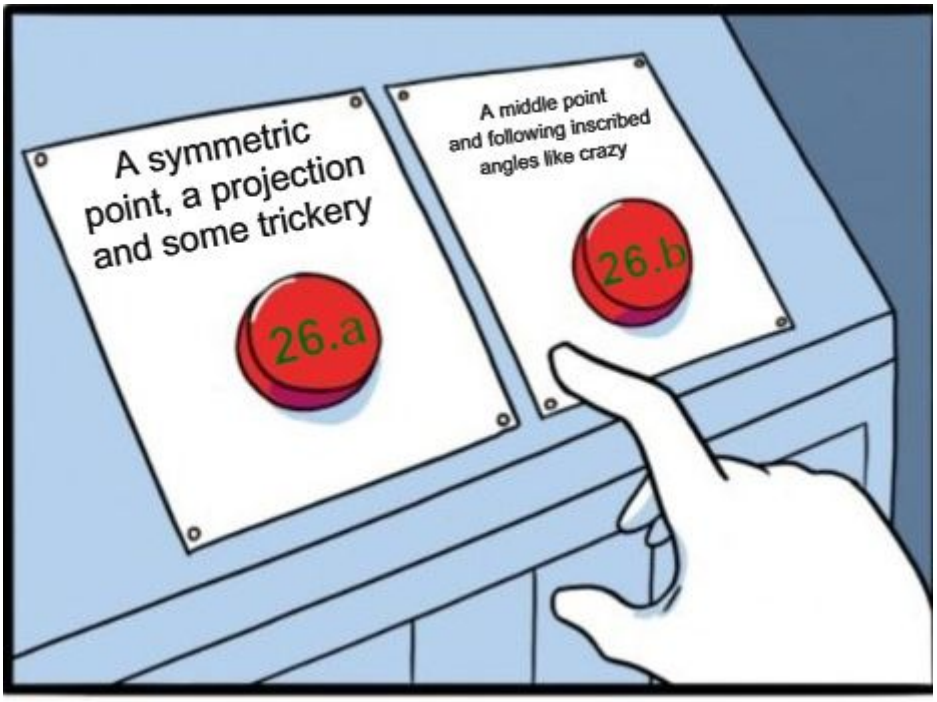
24. Let y be a parabola with two distinct solutions and a leading coefficient 1. Draw a square, using the entire line segment cut by the parabola from the x-axis as one of its sides. Prove that the area of the square equals the discriminant of y.

Proof.
Since the quadratic equation has a leading coefficient of 1, it is: $y(x) = x^2 + bx + c$; $a = 1$
Let D be the discriminant: $D = b^2 + 4ac$.
The side square equals $x_2 - x_1$, where x_1, x_2 are the two distinct solutions. (this follows from the description of the square).
By the formula for the solutions, the difference of the roots is:
$\frac{-b+\sqrt{D}}{2} - \frac{-b-\sqrt{D}}{2} = \frac{-b+b+\sqrt{D}+\sqrt{D}}{2} = \sqrt{D}$.
Thus, its area is the discriminant D as the area of the square is the square of $x_2 - x_1$.

25. In a data set of 5 elements, the mean, mode and median are all $\sqrt{18}$, the range is $\sqrt{50}$ and the IQR (interquartile range) is $\sqrt{2}$. Find all 5 elements.

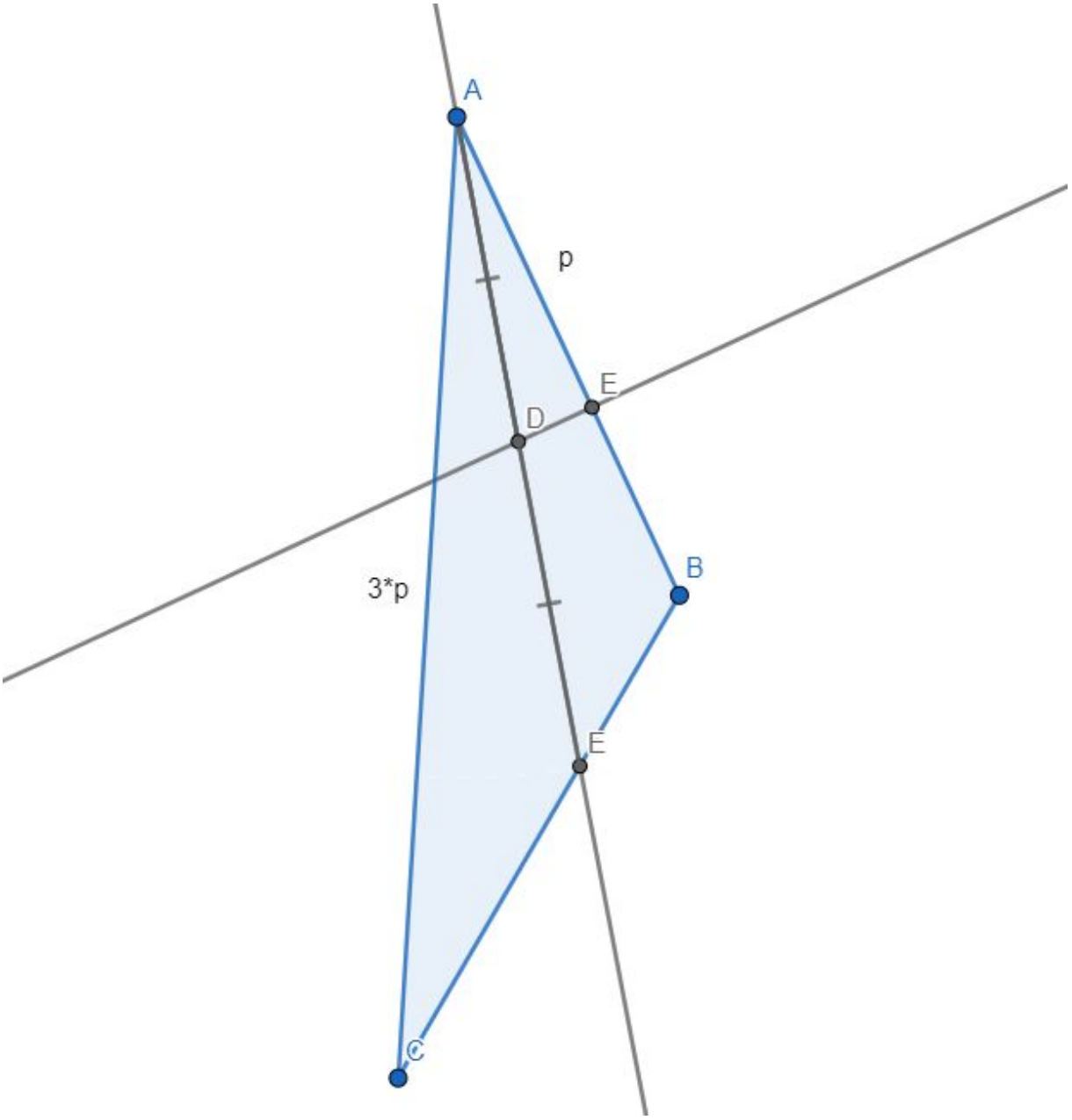
Solution.
Name elements a,b,c,d and e.
Since the median is $\sqrt{18}$, then c= $\sqrt{18}$
$\sqrt{18}$ is also the mode, implying that there is at least one more $\sqrt{18}$.
The range is $\sqrt{50}$, thus e-a= $\sqrt{50}$
d-b= $\sqrt{2}$
By solving the simultaneous equations and dividing each equation by $\sqrt{2}$, one may obtain the data set:
$\sqrt{2}$, $\sqrt{8}$, $\sqrt{18}$, $\sqrt{18}$, $\sqrt{72}$

M5. Solver **either** problem *26.a* **or** *26.b*. You will be given credit for **only one**.



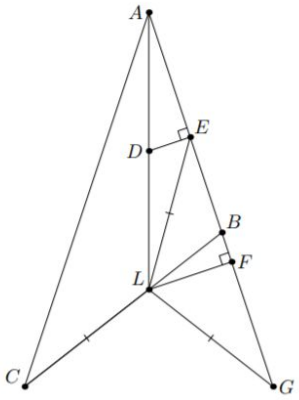
This is a reference to the "Daily Struggle" meme (https://knowyourmeme.com/memes/daily-struggle). Unlike the previous "choose your problem" part, here you should only solve 1 question. You can also earn an extra meme point if you are creative.

26.a. Let AL be the bisector of ΔABC, point D — its middle, E — projection of D on AB. Given that AC=3AE. Prove, that ΔCEL is isosceles.



Proof:

Let F be the projection of L on AB. Then by Thales AE=EF= $\frac{1}{3}$ AC.



Let G be the symmetric point of E with respect to B.

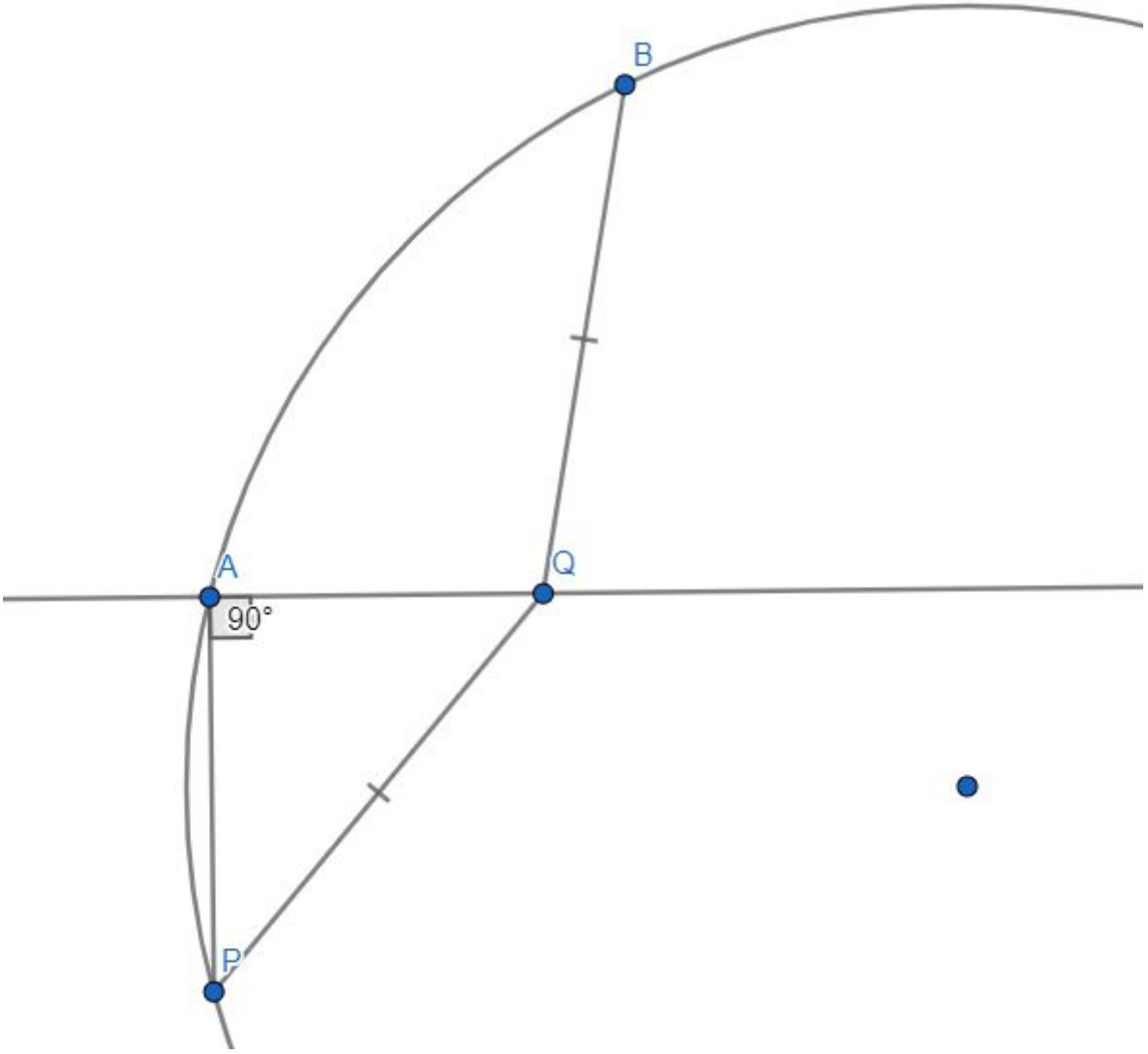
Therefore, AE=EF=FG= $\frac{1}{3}$ AC, so AG=AC.

We could conclude, that ΔACL=ΔALG (AL—common, ∠CAL=∠LAG, AC=AG).

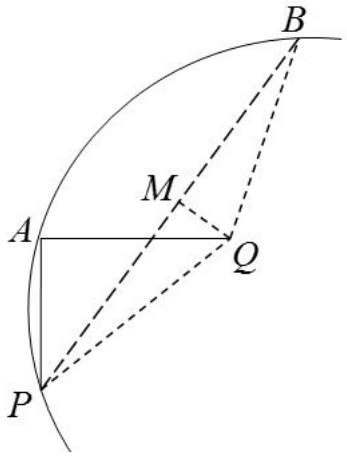
Then CL=LG, but at the same time EL=LG, because LF is median perpendicular, so ΔLEG is isosceles.

Therefore, CL=LG=LE, so CL=LE, thus ΔCLE is isosceles. Let AL be the bisector of ΔABC, point D — its middle, E — projection of D

26.b. Points P, A, B are located on the circle, inside the circle is point Q, such that ∠PAQ=90, PQ=QB, and points P and B are lying on opposite sides of AQ. Prove that ∠AQB-∠PQA=2∠APB.



Proof:



Let M be the middle of PB. Because ΔPQB is isosceles QM⊥PB. Therefore, quadrilateral PAMQ is inscribed. Thus, ∠APM=∠AQM (lean on one arc).

Thus,

$$\angle AQB - \angle AQP = \angle AQM + \angle MQB - \angle AQP =$$

$$= \angle AQM + (\angle MQP - \angle AQP) = 2\angle AQM = 2\angle APM \ ,$$

What was needed to proven.

M6. **(Do this AFTER completing the exam)**

You are at a party organized by the MAT and ECO departments at AUBG and you meet this attractive ECO major you want to impress with your vast mathematic knowledge. The two of you were taking calculus so you go there and say the following:
- (You) [Said with an enquiring and serious voice] All these lectures on derivatives we had to go through really make me want to be one of them. [Pause]
- Oh, why is that? [Generally intrigued]
- So I could lie tangent to your curves.
- [Laughs] Oh, that is sweet, but I have heard that one. You know what, if you can think up of something else that will make me laugh, we can go for dinner some time.

Thankfully, you just watched some videos on Calculus! What more do you say?

AUBG lingo: ECO major – a student majoring in economics; MAT – a student majoring in mathematics.

The point of this "question" is to be a continuation of the last question from the CharisMATHic Analysis exam from 2018. The last "question" there essentially leads to the first part of the conversation here. The goal, obviously, is to have fun and relax at the end of the exam, see if you have seen the previous exams (you might reference it here) and to show how "smooth" you can be after a somewhat cheesy yet smart introduction. Here, you can earn up to 2 meme points: 1 for writing a good follow up to the conversation (it has to be meaningful) and 1 for basing it on mathematics (using what you have learned from the preparatory materials or something you have learned while researching the more humorous homework "problem").