AUBG MultiTalent Quest
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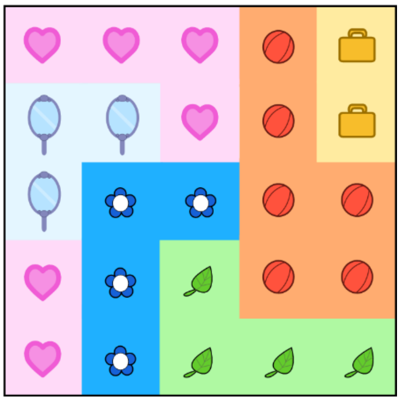
CharisMATHic Analysis
*February 17, 2018*
Problems and Solutions

**1.** If $x$ and $y$ are odd integers, then which of the numbers below must be also odd?

**A)** $xy$ **B)** $x-y$ **C)** $x+y$ **D)** $x+3y$ **E)** $xy+y$

**Answer A.**

**2.** Ann is papering the 5×5 wall using rectangle wallpapers of different sizes. The wallpaper is always placed so as not to go beyond the borders of the wall. She uses a different color (shown with different symbols) for each rectangle of wallpaper. Sometimes, she covers part of the wallpaper with a new rectangle of a different color. In which order did Robyn place the wallpaper?



**A)**  **B)** 

**C)**  **D)** 

**E)** None of the above

**Answer D**. The yellow wallpaper with the briefcases is the only wallpaper that isn't cut off by another one, so that should be the last one. You can see the suitcase cuts off the basketball wallpaper that the basketball wallpaper cuts off the leaf wallpaper, the leaf wallpaper cuts off the flower wallpaper, the flower wallpaper cuts off the mirrors and the mirrors cut off the hearts.

**3.** How many integers $n$ satisfy the condition $3 \le |n| \le 9$?

**A)** 6 **B)** 7 **C)** 10 **D)** 12 **E)** 14

**Answer E.** These are 3, 4, 5, 6, 7, 8, 9 and their opposites.

**4.** If $4^8+4^8+4^8+4^8=4^n$, then $n=$ **A)** 9 **B)** 12 **C)** 16 **D)** 24 **E)** 32

**Answer A.** We have $4.4^8=4^9$.

**5.** Points $M$, $N$ lie on the circumference of a circle $c$. If $MN=6$, find all possible values of the area $A$ of the circle $c$.

**A)** $A=9\pi$ **B)** $A\ge 6\pi$ **C)** $A\ge 9\pi$ **D)** $A>6\pi$ **E)** $A>9\pi$

**Answer C.** If $MN$ is a diameter of $c$, then its radius is 3 and its area is $9\pi$. If $MN$ is not a diameter of $c$, then its radius is greater than 3 and its area is greater than $9\pi$.

**6.** *If* $3x+5y=a$ and $3x-5y=b$, then $xy=$?

**A)** $\frac{a^2-b^2}{15}$ **B)** $\frac{a^2-b^2}{30}$ **C)** $\frac{a^2-b^2}{60}$ **D)** $\frac{a^2-b^2}{120}$ **E)** $\frac{a^2-b^2}{225}$

**Answer C.** If we add the two equations, we get $6x=a+b$, while if we subtract them, we get $10y=a-b$. If we multiply these, we get $60xy=a^2-b^2$.

**7.** The sum of six consecutive integers is 999. The largest one among these integers is:

**A)** 166 **B)** 167 **C)** 168 **D)** 169 **E)** 170

**Answer D.** The average among these integers is $999:6=166.5$, so the numbers are 164, 165, 166, 167, 168, 169.

**8.** Three of the angles in a quadrilateral are equal, while the fourth one is three times smaller. Find the measure of the largest angle in this quadrilateral.

**A)** 54º **B)** 96º **C)** 108º **D)** 120º **E)** 144º

**Answer C.** If the angles are $3x$, $3x$, $3x$, $x$, their sum is $10x=360^\circ$, hence $x=36^\circ$, $3x=108^\circ$.

**9.** In triangle $ABC$ the obtuse angle between the angle bisectors of angles $B$ and $C$ measures 127º. Find the measure of angle $BAC$.

**A)** 53º **B)** 54º **C)** 63º **D)** 74º **E)** 83º

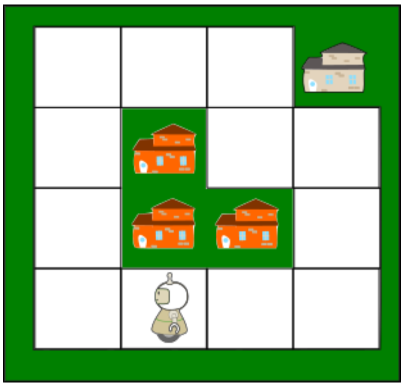
**Answer D.** We have $90^\circ+\frac{1}{2}\angle BAC=127^\circ$, hence $\angle BAC=74^\circ$

**10.** A straight line passes through the points $A(1;8)$, $B(3;2)$ and $C(6;y)$. Find $y$.

**A)** –8 **B)** –7 **C)** –6 **D)** –5 **E)** –4

**Answer B.** The slope of the line is $(2-8)/(3-1)=-3$, so $y=2-3\cdot3=-7$.

**11.** Robot-policeman works in town and every night does its rounds along the street (see the map). Robot-policeman uses only three operations: Forward, Left_Turn (without moving forward), Right_Turn (without moving forward). What is the minimum number of operations needed by robot-policeman to do one round along the street, ending on the same place where it started?



**A)** 12 **B)** 14 **C)** 16 **D)** 18 **E)** 20

**Answer D.** Only the forward operation can move robot-policeman. To change the direction you need two operations, one for turning, and one for moving the robot forward. The road has 12 places, and 6 turns. Hence, 18 operations are needed to make robot-policeman do a full round.

**12.** Sabina has blue beads and twice as many red beads. Some of the red beads are in a jar, while all other beads are in a box. The ratio of red to blue beads in the box is $2:9$. Find the ratio of the quantities of red beads in the box and in the jar.

**A)** $1:5$ **B)** $1:6$ **C)** $1:7$ **D)** $1:8$ **E)** $1:9$

**Answer D.** If in the box there are $2x$ red and $9x$ blue beads, then in total there are $18x$ red marbles, so $16x$ of them are in the jar. The ratio of $2x$ to $16x$ is $1:8$.

**13.** Find the area of the triangle, whose vertices have coordinates $(0;0)$, $(16;30)$ and $(-30;16)$.

**A)** 578 **B)** 608 **C)** 768 **D)** 960 **E)** more than 999

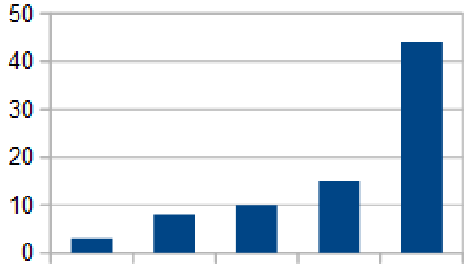
**Answer A.** The area of the rectangle with vertices $(-30;30)$, $(-30;0)$, $(16;0)$ and $(16;30)$ is $46\cdot30=1380$. We need to cut from it two right triangles of total area $30\cdot16=480$ and a right triagle of area $46\cdot14:2=322$. The remaining area is $1380-480-322=578$.

**14.** Let $n$ be the least positive integer such that when $n$ is divided by 5, the remainder is 1; when $n$ is divided by 6, the remainder is 2; and when $n$ is divided by 7, the remainder is 3. What is the remainder when $n$ is divided by 11?

**A)** 5 **B)** 6 **C)** 7 **D)** 8 **E)** 9

**Answer D.** Note that $n+4$ is a multiple of 5, 6, 7 and the least such multiple is 210. Thus $n=206$ and when divided by 11 the remainder is 8.

**15.** Jack was counting the passing cars of different colors and obtained the list: Blue: 10; Red: 44; White: 3; Green: 8; Black: 15. Jack typed the list into a computer program which produced the diagram on the right. However, the columns did not show the colors of the cars. Which color car does the middle column in the table represent?




**A)** Blue **B)** Red **C)** White **D)** Green **E)** Black

**Answer A.** The column in the middle represents the cars which were third most frequent. When we order the five colors according the number of cars, the third value will be 10 – the number of blue cars.

**16.** If $x+y+z=791$, $\frac{x}{y}=\frac{10}{9}$ and $\frac{y}{z}=\frac{6}{25}$, find $y$.

**A)** 108 **B)** 126 **C)** 144 **D)** 162 **E)** 168

**Answer B.** If $y=18k$, then $x=20k$, $z=75k$, $791=x+y+z=113k$, hence $k=7$, $x=126$.

**17.** How many hours are needed to mow a square field of perimeter 1 km, if it is mowed at a rate of 125 sq.m per hour?

**A)** 64 **B)** 128 **C)** 250 **D)** 256 **E)** 500

**Answer E.** The side of the square is 250 m, hence the area is $62500=500\times125$.

**18.** A certain price was increased by 60% and then the new price was decreased by 15%. The same final price would be attained by a single increase of the initial price by:

**A)** 33% **B)** 36% **C)** 39% **D)** 42% **E)** 45%

**Answer B.** $1.6 \times 0.85 = 1.36$.

**19.** Find the perimeter of an isosceles triangle that has one side of length 7 and another of length 17.

**A)** 41 **B)** 34 **C)** 31 **D)** 24 **E)** It cannot be determined

**Answer A.** In order to have a triangle, the third side needs to be larger than the difference of the given two, so its length cannot be 7. Hence it is 17 and the perimeter is $17+17+7=41$.

**20.** The average of the measures of five of the angles of an octagon is 159º. What is the average measure of the remaining three angles of this octagon?

**A)** 95º **B)** 105º **C)** 115º **D)** 125º **E)** 135º

**Answer A.** The sum of these five angles is $5 . 159^\circ=795^\circ$, and the sum of the 8 angles is $6 . 180^\circ=1080^\circ$. The remaining two angles have sum of $1080^\circ-795^\circ=285^\circ$, so their average measure is 95º.

**21.** The average weight of six boys is 59 kg. None of them is heavier than 67 kg, and the weights of each two differ by at least 2 kg. What is the least possible weight (in kg) of one of these boys?

**Answer 39.** The largest possible weights (in kg) of the other 5 boys are 67, 65, 63, 61, 59, so the weight of the heaviest is at most $6 . 59-67-65-63-61-59=39$.

**22.** The area of a circle of radius 27 is greater than the area of a circle of radius 15 by what percent?

**Answer 224.** The ratio of the areas is $\pi27^2/\pi15^2=9^2/5^2=81/25=324/100=1+224/100$.

**23.** To paint the walls and the ceiling of a rectangular room of length 600 cm and width 400 cm one needs 7.3 kg of paint. If one square meter needs 0.1 kg of paint, find the height of the room (in centimeters).
**Answer 245.** The painted area is 73 sq.m, including 24 sq.m for the ceiling. The walls form a rectangle of length 2(4+6) = 20m and area 73–24=49 sq.m, so its height is 49/20=2.45m.

**24.** Point $C$ lies on a circle with diameter $AB$ and area 289π. Find the area of triangle $ABC$ if $AC = 16$.
**Answer 240.** The radius of the circle is 17, so $AB = 34$. By Pythagoras, $AC^2 + CB^2 = AB^2$, hence $BC^2 = 34^2 - 16^2 = 30^2$ and the area is $16.30:2 = 240$.

**25.** Ann put \$1000 for one year in bank with an annual interest of 6%. Ben put \$1000 for one year in bank with an interest of 3% for each six months. What is the difference of their accounts after one year (in cents)?
**Answer 90.** The difference is the interest paid on the interest of 3%.\$1000 = \$30, which is 3%.\$30 = \$0.90.

**26.** Find the largest three-digit number that has remainder 13 when divided by 16 and remainder 8 when divided by 11.
**Answer 877.** If $x$ is the number in question, then $x+3$ is divisible by 16 and by 11, hence by 176. The largest three-digit multiple of 176 is $176.5 = 880$, which yields $x = 877$, while the next multiple (1056) yields a four-digit number.

**27.** A car traveled with an average speed of 96 km/h from Sofia to Blagoevgrad. If on the way back its time was increased by 20%, then its average speed has decreased by how many km/h?
**Answer 16.** The time is multiplied by 6/5, so the speed is multiplied by 5/6. Thus it has decreased by 1/6, and 96/6 = 16.

**28.** A number sequence consists of the numbers 32, 16, 8, 4, 2, 1 and after that these 6 numbers are repeated forever. Find the sum of the 83$^{th}$, 84$^{th}$ and 85$^{th}$ number in this sequence.
**Answer 35.** These numbers coincide with 5$^{th}$, 6$^{th}$ and 1$^{st}$ among these numbers and $2+1+32=35$.

**29.** Mary is six years younger than Tom. Tom is now four times older than Mary. How many years from now will the age of Mary be equal to 80% of the age of Tom?
**Answer 22.** If Mary now is $x$, Tom is $4x = x+6$, hence $x = 2$, $4x = 8$. Their difference is 6 years and it will be equal to 20% of his age when he is $5\times 6 = 30$. This will happen $30-8=22$ years from now.

**30.** Pat has won only 3 of his first 20 games in table tennis. What is the least number of additional games that he needs to play, knowing that he will lose at least 40% of them, in order to have won more than half of his games?
**Answer 75.** If he plays $5x$ games, he will lose at least $2x$ games. Then $3+3x > 17+2x$, so $x > 14$. If $x = 15$, he will have played $5x = 75$ additional games.

**31.** For a moment image that you are called Hippasus and live in the ancient Greek colony Metapontum. You also happen to be a Pythagorean philosopher. One day you decide to find the ratio of which two integers p and q you can represent the number $\sqrt{2}$. However, you seem to be stuck and even consider this to be impossible.
Thus, you try to show that **no** such integers exist such that $\frac{p}{q} = \sqrt{2}$.
Note: Try reasoning about division properties of numbers.
Life advice: After completing the proof, better not share it with your fellow Pythagoreans.

**Explanation**
Whenever you want to show that something does **not** exists or is impossible, it is usually a good idea to do using a proof by contradiction. The outline is as follows: we assume the statement is true and then we attempt to reach a point at which we contradict ourselves which would imply that our original reasoning was incorrect.

**Possible Proof**
Assume $\sqrt{2}$ is rational, i.e. it can be expressed as a rational fraction of the form $\frac{p}{q} = \sqrt{2}$, where p and q are two relatively prime integers. Now, since $\frac{p}{q} = \sqrt{2}$, we have $\frac{p^2}{q^2} = 2$, or $p^2 = 2q^2$. Since $2q^2$. is even, $p^2$. must be even, and since $p^2$ is even, so is p. Let $p = 2r$. We have $4r^2 = 2q^2 \rightarrow 2r^2 = q^2$. But this implies that $q^2$ is even and from that q is also even. However, we arrive at a contradiction – two even numbers cannot be relatively prime. Thus, it is not possible for $\sqrt{2}$ to be a fraction of two integers and thus a rational number.

**32.** You are strolling through Turin in the 1890s and a guy named Peano comes and tells you that he has found a solid foundation for mathematics. But he wants to go on to define his Latine sine Flexione ("Latin without Inflection") language so that people can communicate in a unified language. Thus, needs your help to prove something.

Natural numbers are defined as followed. You have a constant called Z (also know as 0) and a function called successor S (think of this as adding +1 to the number) which gives you the next largest natural number. So, S (Z) is the number after zero (1), S(S(Z)) is 2 and so on. We define addition of natural number as follows:
Let m and n be natural numbers. Define m + n as:

1. If $m = 0$, then m + n = 0 + n = n
2. If $m \neq 0$, thus m = S (k), (k is a natural number), then m + n = S (k) + n = S (k +n)
   [The above is a recursive operation, i.e. we do it until we reach a base case]

Using the above definition show that m + n = n + m.

Note: The above definition is inductive so try an inductive proof. A reminder, an inductive proof means that you must have a base case such that it holds for zero and a way to show that it holds for the next element (the plus one).
Note 2: You can assume that m + 0 = m as well (even though this also needs a proof).

**Explanation**

The above construction of natural numbers has fundamentally two different patterns or kinds of numbers: zero (Z) and the next element of a natural number (S (k), where k is some number). To prove something about these natural numbers we must show that it holds for both kinds. Thus, we need to show that n + m = m + n (using the definition given) holds for any combination of n and m for both Z and S (k). We will also use the second note (because without it, we must prove a corollary and as some people would say "Ain't Nobody Got Time for That!"), so that we assume that m + 0 = 0. This is key as the definition of plus in the base case is 0 + m = m. When we have the two base cases, we can simply assume the "inductive hypothesis" and commute the two numbers, i.e. k + m = k + m. With induction we use the thing we are proving for smaller cases (i.e. we try to prove something about k+1 but we assume that it is true for k).

**Possible Proof**

Directly expand the meaning of plus:

$$\begin{aligned} n + m &= if(n = 0) &&\rightarrow 0 + m = m = m + 0 \\ &= if(m = 0) &&\rightarrow n + 0 = n = 0 + n \\ &= if\left(n = S(k)\right) &&\rightarrow S(k) + m = S(k + m) \end{aligned}$$

The first and second cases are by Note 2 and the definition of addition for zero (but in reverse) and the third case is just direct application of the definition of addition Consider what we have shown. The first two cases say that whenever we have a zero at any one place, we can commute the argument. Thus, we have a base case and we can assume the inductive hypothesis so that we can commute two natural numbers during addition thus the third case becomes $S(k) + m = S(k + m) = S(m + k)$. Now, we wish to show is that $S(m + k) = m + S(k)$, which will complete the above proof. When m = 0, the definition is trivial (as before and thus our base case). When $m = S(l)$, we apply our new inductive hypothesis and get $S(S(l) + k) = S\left(S(l + k)\right) = S\left(l + S(k)\right) = S(l) + S(k) = m + S(k)$ (Here we just use the definition of addition and the inductive hypothesis. Now, using this proof, we have: $S(k) + m = S(m + k) = m + S(k)$ which completes the proof.

**33.** Consider yourself a young mathematics student at Göttingen in the 1920s. You meet Emmy Noether and she thinks you might be a good candidate to become one of her student assistants on this new approach to doing algebra, so she decides to test your abstract reasoning.

Imagine you have a set of values (x, y, z, etc.), an operation *op* which takes two elements and produces a third element (*op(x, y) = z*), an operation *inv* which takes an element and produces a different element (*inv(x) = y*) and a constant (an element of the set) called *e*. In this set, the following axioms hold:

1. *op(x, op(y, z)) = op(op(x, y), z)* (associativity, i.e. you can reorder braces)
2. *op(x, e) = op(e, x) = e* (identity, i.e. *e* does not change anything)
3. *op(x, inv(x)) = op(inv(x), x) = e* (cancellation, i.e *inv(x)* is the opposite of x)

Using the above, show that for any elements in this set *a* and *b*, the following holds true

- *inv(op(a,b)) = op(inv(b), inv(a))*

**Explanation**

The easiest way to approach this problem is just gradually apply either the operation, the inverse and the associativity, identity and cancellation property. You essentially start with the equation of the cancellation property expanded for the thing we are looking for. A way to help understand what is going on is to replace *op* with + and *inv* with –(…). Thus, *inv*(*op*(a,b)) = *op*(*inv*(b), *inv*(a)) is translated to –(a + b) = -b – a (which is obviously correct). However, there is no rule saying that you can commute the two elements so you must not change the places of two consecutive elements.

**Possible Proof**

Begin with the equation for inverse, i.e. the cancellation property above:

$$op\left(inv(op(a,b)),op(a,b)\right)=e$$

By 3), this is always true. Now apply the element *inv*(b) to both sides:

$$op(op\left(inv(op(a,b)),op(a,b),inv(b)\right)=op(e,inv(b))$$

Now, use reorder the last operation brackets so that b and *inv*(b) are together. Also reduce the right side with the property of *e*.

$$op(inv(op(a,b)),op\left(a,op(b,inv(b))\right)=inv(b)$$

We can apply 3) to the term $op(b,inv(b))=e$, thus replace it with $e$ and then once again reduce to just leave the a:

$$op(inv(op(a,b)),a)=inv(b)$$

We do the exact same thing with *inv*(a) as we did with *inv*(b) (apply the operation on both sides to the right using *inv*(a)) to arrive at:

$$op(op(inv(op(a,b)),a),inv(a))=op(inv(b),inv(a))$$

We, once again use 1) to associate the operations and then reduce the $op(a,inv(a))$ part. This way, we arrive at the final version:

$$op(inv(op(a,b),op(a,inv(a)))=op(inv(b),inv(a))$$
$$op(inv(op(a,b),e)=op(inv(b),inv(a))$$
$$inv(op(a,b)=op(inv(b),inv(a))$$

And, with this, we arrive at the equality we wish to prove.

**Note:** Try translating all the op and inv calls with addition (a+b) and negation (-a) but only use the rules above. Write down 0 instead of *e*. The resulting proof should be simpler to read.

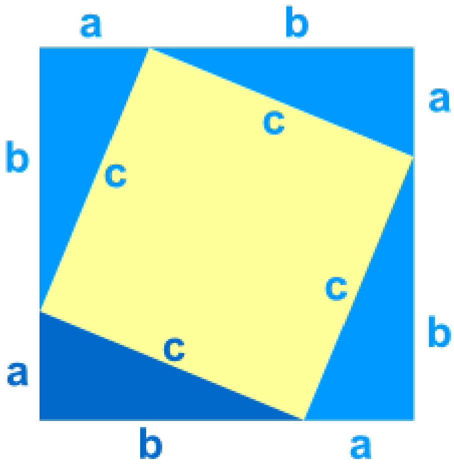
**34.** February 15, 2018, you tell your grandma that you cannot come and eat because you have a mathematics exam in two days and you want to revise the proofs around the Pythagorean theorem. She mentioned she has forgotten the theorem. Now you have to explain the theorem to your grandma.

Give a visual proof of the Pythagorean theorem $a^2 + b^2 = c^2$. Try to reason about how it relates to a right-handed triangle and the area it expresses.

**Possible Proof**

Just drawing something like this is a valid visual proof, as long as you also show the relation between the different areas. The whole square has area $(a + b)^2 = a^2 + 2ab + b^2$. Each of the 4 triangles has area $\frac{ab}{2}$, so the area of all triangles is $2ab$. Once you substract the area of the triangles from the area of square, you get $a^2 + 2ab + b^2 - 2ab = a^2 + b^2$. But this area is exactly the areas of the square formed by the hypothenuses of the triangles and it equals $c^2$.

Thus, $a^2 + b^2 = c^2$.



**35.** You are in the club and this guy challenges you to prove that $x^n + y^n = z^n$ has no integer solutions for $n > 2$ and $x, y, z \neq 0$. How do you respond?

**Explanation**

As explained several times, this “question” and the next one, are simply in the test for a little bit of fun as well as to take your mind of the test and ease you in for the debate and the social challenge disciplines.

The person in the picture is Pierre de Fermat, a famous French mathematician from the 17th century (whose main occupation was law but that is beside the point now). He has had substantial influence on several mathematical disciplines during his time, but you can read about that on your own. The proposition in the description is know as Fermat’s Last Theorem, a famous theorem in number theory. Fermat first stated this theorem in margin of a copy of Arithmetica (a book on mathematics) but he never formally proved it. The first successful proof of this theorem surfaced in 1994, by Andrew Wiles (so, a long time after it was stated).

Now, on to the idea of it all. Obviously, one part is a reference to the meme “You are in the club and this guy…" which to some extent expects you to answer in some witty way but also predisposes you to the impossibility of an adequate response in the literal sense. The other part is that Fermat himself wrote about the theorem that “I have a truly marvelous demonstration of this proposition which this margin is too narrow to contain” but he never got around to writing it. And, it is impossible for us to know now if he truly had a proof! He was a great mathematician, but he could have also just been messing with the readers.

Anything written here is acceptable, but the most hilarious answers would have been in the spirit of what Fermat wrote about his mysterious proof. I would have personally gone with:

“I have a truly incredible proof of this proposition which the music in this club is too loud for me to explain.”

(Fun fact. A finalist **actually** wrote something almost the same! You know who you are, and you made my day!)

**36.** You are at party organized by the MAT and ECO departments at AUBG and you meet this fine ECO major you want to impress with your vast mathematic knowledge. What do you say?

AUBG lingo translation ECO major – a person majoring in economics; MAT – short for anything mathematic related.

**Explanation**

This should be relatively obviously – you are supposed to either present some pickup line or a short dialogue aimed at starting a conversation on the right foot. Obviously to achieve your goal, you should think of something witty and funny on the spot, which must also be appropriate for the current situation (i.e. be based on mathematics and/or economics). Honestly, this was inspired by the debates discipline in which you must also think of something on the spot to impress the judges with your conversation presence (but mostly replace funny with logical and/or inspiring).

In addition to all the above, when you start your bachelor's degree in any university (especially true for AUBG) you will experience situations like this on numerous occasions. You want to attract or befriend someone you have never met at an event on campus, regardless of your reasons, (one is pretty obvious… but you get my point) and you know what the person is most likely interested in – after all they are at the event. Thus, your most adequate course of actions is to say something relevant for or referring to the event.

For the given situation, I would honestly go for one of the following (both on the funny side so proper delivery is important to have a chance at adequately continuing the conversation):

Witty, mathematical, but depending on the delivery it could be cheesy (usually not good):

- [Said with an enquiring and serious voice] All these lectures on derivatives we had to go through really make me want to be one of them. [Pause]
- Why is that? (Or hopefully something similar)
- So, I could lie tangent to your curves.

Somewhat traditional, economics related, less uncertainty in the response:

- [Cool, collected] Hey, is it me or has the supply of stunning ECO majors increased, because the market forces are pushing me to start a conversation with you.